Completeness of multiplicative Ehresmann connections on fibred Lie groupoids

Matthijs Lau

Institute of Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen

Summary

A connection on a surjective submersion gives rise to a notion of parallel transport. On surjective submersions, completeness of connections translates to the local triviality over the base. Similar problems have been solved for Lie groupoid extensions, surjective submersive Lie groupoid morphisms that cover the identity.
We extend this result further to surjective submersions of Lie groupoids covering arbitrary maps, and relate completeness to local triviality on the kernel bundle and object maps. Additionally, we give new geometrical tools to check for the completeness of a connection. Based on work by Fernandes, Mărcuț (2023), and del Hoyo (2016).

Known results

We have the following results:

1. A bundle of Lie groups admits a complete multiplicative connection if and only if it is locally trivial as a bundle of groups.
2. A connection on a Lie groupoid extension is complete if and only if the induced connection on the kernel is complete.

Can we generalise these ideas to more general morphisms?

Fibred Lie groupoids

- A *fibred Lie groupoid* is a Lie groupoid morphism $\pi\colon \mathcal{G} \to \mathcal{H}$ such that π is a surjective submersion. Remark that π may cover some non-trivial surjective submersion $\pi_0\colon M \to N$.
- A *bundle of Lie groupoids* is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a surjective submersion $p\colon M \to B$ such that $p \circ s = p \circ t$.
  - Bundles of Lie groupoids correspond to fibred Lie groupoids $p\colon \mathcal{G} \to B$, where $B \rightrightarrows B$ is the identity groupoid.
  - Given a fibred Lie groupoid $\pi\colon \mathcal{G} \to \mathcal{H}$, with $\mathcal{H} \rightrightarrows N$, the restriction of π to the kernel $\ker \pi = \pi^{-1}(N)$ defines a bundle of Lie groupoids.
- An *isomorphism of bundles of Lie groupoids* between $p\colon \mathcal{G} \to B$ and $q\colon \mathcal{H} \to Q$ is a Lie groupoid isomoprhism $\psi\colon \mathcal{G} \to \mathcal{H}$ such that $p = q \circ \psi$.
- A bundle of Lie groupoids $p\colon \mathcal{G} \to B$ is called *locally trivial* if there exists a cover $\{U_\alpha\}$ of B such that $p\colon p^{-1}(U_\alpha) \to U_\alpha$ is isomorphic to $\mathrm{pr}_1\colon U_\alpha \times \mathcal{F} \to U_\alpha$ as bundles of Lie groupoids.

Multiplicative connections

- A *multiplicative Ehresmann connection* on a fibred Lie groupoid $\pi\colon \mathcal{G} \to \mathcal{H}$ is a distribution $E \subset T\mathcal{G}$ which is multiplicative, i.e. it is a subgroupoid of $T\mathcal{G} \rightrightarrows TM$, and $E \oplus \ker d\pi$.
- Geometrically, this is the correct notion of a multiplicative connection as it lifts paths multiplicatively. Some smooth paths $\gamma_1, \gamma_2\colon [0,1] \to \mathcal{H}$ and $g_i \in \pi^{-1}(\gamma_i(0))$ such that
  1. the paths are composable: $s \circ \gamma_1 = t \circ \gamma_2$,
  2. the lifts are composable: $s(g_1) = t(g_2)$,

  then $s \circ \tilde{\gamma}_{1g_1} = t \circ \tilde{\gamma}_{2g_2}$

Analytical tools

Analytically, a connection on $\pi\colon M \to B$ is called *complete* if any lift $\tilde{\gamma}_x$ of a path $\gamma\colon I \to B$ is defined on I. Instead, this can be checked geometrically.

Lemma

Let E be an Ehresmann connection on $\mathrm{pr}_1\colon B \times F \to B$ and suppose that there exists $S \subset F$ such that

- for $(b,s) \in B \times S$ it satisfies $E_{(b,s)} = T_bB$;
- the connected components of $F \backslash S$ are relatively compact;

then E is complete.

This generalises to locally trivial fibre bundles by requiring suitable S_α on each trivialization.

Multiplicative main theorem

Families of Lie groupoids are the multiplicative versions of surjective submersions. The following partially replicates the relation between fibre bundles and complete connections in the multiplicative setting.

Theorem

A family of Lie groupoid $p\colon \mathcal{G} \to B$ with a complete multiplicative connection is locally trivial. If $\mathcal{G}$ is s-proper, then the converse holds as well.

Additionally, as multiplicative Ehresmann connections on a family of Lie groupoids define a connection on the map of objects. The completeness of these two connections is then essentially related under some mild conditions.

Proposition

Let $\pi\colon \mathcal{G} \to B$ be a family of Lie groupoids and E a multiplicative Ehresmann connection:

- If E is complete, then so is E_0.
- If E_0 is complete and $\mathcal{G}$ is s-connected, then E is complete.

Relation with fibred Lie groupoids

While fibred Lie groupoids do not admit local trivializations, we can consider their kernel bundle. The non-local triviality of this bundle then becomes an obstruction to the existence of complete connections on the total fibred Lie groupoid.

Theorem

For a fibred Lie groupoid $\pi\colon \mathcal{G} \to \mathcal{H}$ and a multiplicative connection E:

- $E^{\ker} = E \cap T\ker\pi$ is a multiplicative connection on $\pi\colon \ker\pi \to N$.
- If E is complete, then so is $E^{\ker}$ as well.
- If $E^{\ker}$ is complete and $s\colon \pi^{-1}(h) \to \pi_0^{-1}(s(h))$ is surjective, then E is complete.

Again, this gives possible obstructions for the existence of complete connections in terms of local triviality of the kernel bundle.