Multiplicative Ehresmann connections for Lie groupoid fibrations

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Summary

We extend Ehresmann connections on a surjective submersion to the multiplicative setting.

- These objects generalize simultaneously: classical Ehresmann connections, linear connections, principal connections, and multiplicative Ehresmann connections for Lie groupoid extensions.
- The existence is not guaranteed in general, only for specific classes of morphisms (Theorem 2).
- Under suitable assumptions, the completeness can be reduced to the kernel bundle or the base connection (Theorem 3).

This poster is based on [LM26].

Ehresmann connections

- An ***Ehresmann connection*** on a surj. subm. $\pi\colon M \to B$ is a splitting for $\ker T\pi$, denoted by Ver. These manifest in three flavours: complements, horizontal lifts, and connection forms.
- These can also be viewed in terms of their horizontal lifting map on germs.

$$\mathrm{hl}\colon \mathscr{C}_0^\infty(\mathbb{R}, B) \times_B M \to \mathscr{C}_0^\infty(\mathbb{R}, M), \qquad ([\gamma], x) \mapsto [\tilde{\gamma}_x].$$

  If it extends to paths, say $\mathrm{hl}\colon C^\infty(\mathbb{R}, B) \times_B M \to C^\infty(\mathbb{R}, M)$, the connection is called ***complete***.
- The existence of complete connections is directly tied to the geometry of the submersion:

Theorem 1 [dH16]

The existence of a complete Ehresmann connection is equivalent to local triviality.

Fibration types

- Fibrations of Lie groupoids should incorporate both the smooth and algebraic structure. For the smooth structure, surjective submersions suffice; however, surjective groupoid morphisms do not admit an isomorphism theorem.
- Fibrations adapted to groupoids can be found in [Mac05]. To us, the following are of interest.

Definition 1

A Lie groupoid morphism $\pi\colon \mathcal{G} \to \mathcal{H}$ over $\pi_0\colon M \to N$ is called a

1. ***Lie groupoid fibration***, if the following map is a surjective submersion:
$$\pi^!\colon \mathcal{G} \to \pi_0^!\mathcal{H} := \mathcal{H}_{\boldsymbol{s}} \times_{\pi_0} M, \qquad g \mapsto (\pi(g), \boldsymbol{s}(g));$$
2. ***uniform Lie groupoid fibration*** (resp. ***Morita fibrations***), if the following map is a surjective submersion (resp. diffeomorphism):
$$\pi^*\colon \mathcal{G} \to \pi_0^*\mathcal{H} := M_{\pi_0} \times_{\boldsymbol{t}} \mathcal{H}_{\boldsymbol{s}} \times_{\pi_0} M, \qquad g \mapsto (\boldsymbol{t}(g), \pi(g), \boldsymbol{s}(g));$$
3. ***family of Lie groupoids***, if $\mathcal{H} = N \rightrightarrows N$ is the unit groupoid.

- These classes are related by the following implications

$$\begin{tikzcd} & \text{family} \arrow[dl, Rightarrow] & & \\ \text{surj. subm.} & \text{fibration} \arrow[l, Leftarrow] & \text{uniform fibration} \arrow[l, Rightarrow] & \text{Morita fibration} \arrow[l, Rightarrow] \end{tikzcd}$$

Remark that any surjective submersion of Lie groupoids defines a family of Lie groupoids by restricting to the kernel.

Multiplicative Ehresmann connections

- To extend the notion of an Ehresmann connection to the multiplicative setting, we remark that a surjective submersion of Lie groupoids induces a short exact sequence of $\mathscr{V}\!\mathscr{B}$-groupoids, not merely vector bundles. Using the splitting lemma for $\mathscr{V}\!\mathscr{B}$-groupoids, we can directly translate Ehresmann connection to the multiplicative setting:

Definition 2: Multiplicative Ehresmann connections

A ***multiplicative Ehresmann connection (MEC)*** on π is a splitting of the vertical bundle in the category of $\mathscr{V}\!\mathscr{B}$-groupoids; in particular, we have a 1-1 correspondence between:

$$\left\{\begin{matrix}\text{Multiplicative Ehresmann connections} \\ \mathscr{V}\!\mathscr{B}\text{-subgroupoids } E \subset T\mathcal{G} \text{ s.t.} \\ T\mathcal{G} = E \oplus \mathrm{Ver}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Multiplicative horizontal lifts} \\ \mathscr{V}\!\mathscr{B}\text{-groupoid morphisms } \mathrm{hor}\colon \pi^*T\mathcal{H} \to T\mathcal{G} \\ \text{s.t. } T\pi \circ \mathrm{hor} = \mathrm{id}_{\pi^*T\mathcal{H}}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Multiplicative connection form} \\ \text{Multiplicative forms } \omega \in \Omega^1(\mathcal{G}; \mathrm{Ver}) \\ \text{s.t. } \omega|_{\mathrm{Ver}} = \mathrm{id}_{\mathrm{Ver}}\end{matrix}\right\}$$

- MECs generalize many classical notions of connection: Ehresmann connections, linear connections, principal connections, and MECs for Lie groupoid extensions (cf. [FM23])
- Classically, connections are tied to the fibrewise geometry through their parallel transport. MECs admit a similar description, making them the multiplicative notion of a connection.

Proposition 1: Interpreting multiplicativity

Let $\pi\colon \mathcal{G} \to \mathcal{H}$ be a surjective submersion of Lie groupoids over $\pi_0\colon M \to N$, and suppose E and E_0 are connections for π and π_0, respectively. Then $E \rightrightarrows E_0$ is a MEC if and only if the E-horizontal paths define a subgroupoid $C^\infty(\mathbb{R}, \mathcal{G})^E \rightrightarrows C^\infty(\mathbb{R}, M)^{E_0}$ of the current groupoid $C^\infty(\mathbb{R}, \mathcal{G}) \rightrightarrows C^\infty(\mathbb{R}, M)$.

Existence

- A Lie groupoid fibration does not admit a MEC in general, or even under topological restrictions like compactness. This can be deduced from the following example:

Counterexample

Let $G \ltimes M \rightrightarrows M$ be an action groupoid of a connected Lie group. The existence of a MEC on the following projection

$$\pi\colon G \ltimes M \to G, \qquad (g, x) \mapsto g,$$

implies that the action is trivial.

- For certain classes of morphisms, existence is guaranteed:

Theorem 2: Cases for existence

MECs always exist for:

- Morita fibrations,
- proper uniform Lie groupoid fibrations,
- locally trivial families of Lie groupoids, and
- proper families of Lie groupoids.

- **Next step**: Find the obstruction class in cohomology to the existence problem. Combining the approach in [Gra25] and multiplicative connection forms.

Completeness

- Clearly, if $E \rightrightarrows E_0$ is a MEC on π and E is complete, then so is E_0. The converse does in general not hold; however, there is an intermediate obstruction to completeness.

  We consider $E^{\ker} = E \cap T \ker\pi$ as a MEC on the family of Lie groupoids $\pi\colon \ker\pi \to N$. The completeness of E can be reduced to $E^{\ker}$ or E_0.

Theorem 3: Relations for completeness

The following implications hold regarding completeness

$$E \text{ is complete} \Longrightarrow E^{\ker} \text{ is complete} \Longrightarrow E_0 \text{ is complete}$$

$$\pi \text{ is a fibration} \qquad\qquad \ker\pi \text{ is source-connected}$$

- In the case of families of Lie groupoids, we can find an extension of the classical result in Theorem 1, relating the existence of complete MECs to the local triviality as a family of Lie groupoids.

Theorem 4: Case of families

Let $\pi\colon (\mathcal{G} \rightrightarrows M) \to N$ be a family of Lie groupoids, where $\mathcal{G}$ is a source-proper Lie groupoid. Then π is a locally trivial family of Lie groupoids if and only if it admits a complete MEC.

References & Acknowledgements

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